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Link between the strong-coupling and the weak-coupling asymptotic perturbation expansions for the quartic anharmonic oscillator

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Abstract. For eigenvalues $E(\beta)$ of the Hamiltonian of the quartic anharmonic oscillator we give a representation which reproduces two well known limiting cases from the theory of the anharmonic oscillator expansions: the weak-coupling asymptotic perturbation and the convergent strong coupling. We give an estimation of the radius of convergence of the strong-coupling expansion for the ground-state eigenvalue.

1. Introduction

The model systems which are exactly unsolvable but at the same time tractable in the sense that their properties can be studied more or less detailed, always play a special role in quantum mechanics and quantum field theory. On the one hand, by studying such model systems one obtains deeper insight into the properties of more complex systems. On the other hand, such systems play an important role in the development of different numerical computational schemes.

A well known example of the one-dimensional exactly unsolvable quantum mechanical system is provided by the anharmonic oscillator with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{x^2}{2} + \beta\frac{x^4}{2}. \tag{1}$$

This system has been studied by many authors. The existing literature on the anharmonic oscillators is extremely vast, we shall mention only a few works. A review of the work on this subject before 1980 can be found in the paper by Killingbeck (1977).

The properties of the eigenvalues of the Hamiltonian (1) as functions of the anharmonicity constant β were studied in details in the works by Simon (1970) and Bender and Wu (1969, 1973). It was shown by Simon (1970) that if one considers the operator $\beta x^2/2$ in equation (1) as a perturbing operator and applies formally the Rayleigh–Schrödinger perturbation theory, then the resulting perturbation expansion

$$E(\beta) \sim \sum_{n=0}^{\infty} E_n\beta^n \tag{2}$$

is an asymptotic one and can be used in every sector $|\arg(\beta)| < \theta < 3\pi/2$. A convergent expansion for an eigenvalue $E(\beta)$ can be obtained if one considers the opposite case of the large β-values. It is known (Simon 1970), that $E(\beta)$ can be represented as the following converging series in powers of $\beta^{-\frac{2}{3}}$ (the so-called strong-coupling expansion)

$$E(\beta) = \beta^{\frac{1}{3}} \sum_{m=0}^{\infty} c_m \beta^{-\frac{2m}{3}}. \tag{3}$$

In view of the radius of convergence of this expansion, it was shown (Simon 1970) that it is a finite number, i.e. the series (3) cannot converge for all β.

The studies of the system described by the Hamiltonian (1) contributed considerably to the development of different numerical schemes, especially the techniques of the summation of the divergent series (Arteca *et al* 1990), such as Padé approximants technique (Simon 1970, 1982, Čižek and Vrscay 1982), or Borel summation procedure (Hirsbrunner and Loeffel 1975). On the basis of the strong-coupling expansion (3) different efficient computational schemes, such as the so-called renormalized strong-coupling expansion (Vinette and Čižek 1991, Janke and Kleinert 1995, Weniger 1996a) have been constructed.

The weak-coupling asymptotic perturbation expansion (2) is well studied. The recurrence relation proposed in the paper by Bender and Wu (1973) allows us to compute easily the coefficients E_n up to quite a high order. In this work the law of the asymptotic large-n behaviour of the coefficients E_n was found, which in case of the ground-state eigenvalue reads

$$E_n \sim (-1)^{n-1} \sqrt{\frac{6}{\pi^3}} \Gamma(n+\tfrac{1}{2}) (\tfrac{3}{2})^n \left(1 - \frac{a_1}{n} - \frac{a_2}{n^2} - \frac{a_3}{n^3} \ldots\right) \tag{4}$$

when $n \to +\infty$. For the first two coefficients a_i on the right-hand side of the equation (4) one has (Bender and Wu 1973): $a_1 = \frac{95}{72}$, $a_2 = 20\,099/10\,368$.

In view of the strong-coupling expansion (3), the coefficients c_n have been calculated in a number of works (Fernandez 1992, Fernandez and Guardiola 1993, Janke and Kleinert 1995, Weniger 1996b). Numerical calculations (Janke and Kleinert 1995, Weniger 1996b) indicate that the series (3) converges for β as small as 0.2.

The fact that the two expansions (2) and (3) are somehow connected has been known for a long time. It is known (Arteca *et al* 1990) that by means of a suitable normalization process the coefficients of the strong-coupling expansion can be computed provided that the weak-coupling coefficients are known. Examples of such procedures are presented in the papers by Janke and Kleinert (1995) and Weniger (1996b). In the latter paper, x^6 and x^8 anharmonicities have also been considered.

In this paper we obtain a representation for $E(\beta)$ from which both strong-coupling and weak-coupling expansions can be obtained as two limiting cases. Our consideration will rely on the observation that the regions of applicability of the expansions (2) and (3) overlap, i.e. one can choose β so that $E(\beta)$ can be computed both by means of the weak-coupling asymptotic expansion (2) and the convergent strong-coupling expansion (3).

2. Theory

2.1. A representation for $E(\beta)$

The function $E(\beta)$ is known to be a regular analytic function of β in the complex β-plane with a cut drawn along the negative real axis (Simon 1970, Bender and Wu 1973). $E(\beta)$ has two singular points: $\beta = 0$ and $\beta = \infty$. Consider the function $G(\beta) = \frac{E(\beta)}{\beta}$. From the known analytic properties of $E(\beta)$ if follows that $G(\beta)$ is an analytic function of β in the complex β-plane cut along the negative real axis, regular everywhere except $\beta = 0$

and $\beta = \infty$. From equation (3) one can see that $G(\beta)$ tend uniformly to zero when $\mathrm{Re}\,\beta \to +\infty$. It is known (Markushevitch 1968) that these conditions are sufficient to guarantee that for $\mathrm{Re}\,\beta > 0$ the function $G(\beta)$ can be represented as a Laplace transform of a certain function $f(t)$ and, hence for $\mathrm{Re}\,\beta > 0$ one can write

$$E(\beta) = \beta \int_0^\infty f(t) \mathrm{e}^{-\beta t}\,\mathrm{d}t. \tag{5}$$

The analogous representation has been considered in different context by Delabaere *et al* (1997).

The function $f(t)$ can be expressed through the coefficients of the strong-coupling expansion (3) as follows. Inverting the Laplace transform on the right-hand side of the equation (5) one obtains for $f(t)$

$$f(t) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{E(\beta)}{\beta} \mathrm{e}^{\beta t}\,\mathrm{d}\beta. \tag{6}$$

The integral in equation (6) can be taken along any straight line $\mathrm{Re}\,\beta = c > 0$. The strong-coupling expansion converges for sufficiently large $|\beta|$-values. Therefore, by choosing the parameter 'c' to be a sufficiently large positive number, one can choose the contour of integration in formula (6) so that the strong-coupling series converges everywhere on the contour. Substituting expansion (3) for $E(\beta)$ under the integral sign in equation (6) and performing term-by-term integration with the help of the known formula (Abramovitz and Stegun 1964)

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi\mathrm{i}} \int_C x^{-z} \mathrm{e}^x\,\mathrm{d}x \tag{7}$$

where the contour of integration C is a straight line $\mathrm{Re}\,x = \text{constant} > 0$, one obtains for the function $f(t)$

$$f(t) = \sum_{m=0}^{\infty} \frac{c_m}{\Gamma(\frac{2m+2}{3})} t^{\frac{2m-1}{3}}. \tag{8}$$

The coefficients c_m in equation (8) are the coefficients of the strong-coupling expansion (3). It is easy to see that series (8) converges for all t. Indeed, the strong-coupling expansion has a non-zero radius of convergence. In equation (8) by making a substitution $t^{\frac{2}{3}} = u$ one has, according to the Hadamard formula

$$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = \frac{1}{R} \tag{9}$$

where R is the radius of convergence of the series $\sum c_n u^n$. Applying the Hadamard formula to the series $\sum c_n u^n / \Gamma(\frac{2m+2}{3})$ and using asymptotic expression for the gamma function, one can see that this series converges for all u.

The next question is how to obtain the asymptotic expansion (2) from the integral representation (5). At this stage we have to make an assumption about the function $f(t)$ from equation (8). A rather convincing numerical proof that this assumption is valid will be given below. Let us suppose that the function $f(t)$ can be represented as

$$f(t) = E_0 + g(t) \tag{10}$$

where E_0 is the zero-order coefficient of the perturbation expansion (2) and the function $g(t)$ decays sufficiently rapidly when $t \to \infty$ so that all the integrals $\int_0^\infty g(t) t^m\,\mathrm{d}t$ for $m > 0$ exist. If assumption (10) is valid then, expanding exponential function under the integral sign in the equation (5) and integrating the series obtained term-by-term one obtains for

Table 1. The function $f(t)$ and its asymptotic approximations $f^{(k)}(t)$.

t	$f(t)$	$f^{(2)}(t)$	$f^{(3)}(t)$	$f^{(4)}(t)$
1	0.578 569 30	0.652 309 09	0.406 180 22	0.114 012 06
5	0.509 306 58	0.520 235 15	0.505 611 42	0.497 848 17
10	0.502 190 31	0.504 459 01	0.502 180 37	0.501 325 02
15	0.500 735 65	0.501 397 03	0.500 814 13	0.500 635 47
20	0.500 293 71	0.500 525 15	0.500 335 39	0.500 285 02
25	0.500 130 55	0.500 221 78	0.500 150 10	0.500 133 09
30	0.500 062 55	0.500 101 74	0.500 071 72	0.500 065 22
32	0.500 047 40	0.500 075 87	0.500 054 20	0.500 049 65
34	0.500 036 24	0.500 057 10	0.500 041 27	0.500 038 05
36	0.500 027 97	0.500 043 32	0.500 031 66	0.500 029 35
38	0.500 021 85	0.500 033 12	0.500 024 44	0.500 022 77
39	0.500 019 45	0.500 029 04	0.500 021 52	0.500 020 10
40	0.500 017 42	0.500 025 50	0.500 018 99	0.500 017 76

$E(\beta)$ the series analogous to (2), where the coefficients E_n with $n > 0$ are given by the following formula

$$E_n = \frac{(-1)^{n-1}}{(n-1)!} \int_0^\infty g(t) t^{n-1}\,\mathrm{d}t. \tag{11}$$

It is easy to see that the above described procedure (expansion of the exponential function under the integral sign and further term-by-term integration), yields an asymptotic series, i.e. if $E_N(\beta)$ is the Nth partial sum of the series, then $|E(\beta) - E_N(\beta)| = \mathrm{O}(|\beta|^{N+1})$ when $\beta \to 0$ remaining in the half-plane $\mathrm{Re}\,\beta > 0$. We shall not dwell upon the proof of this fact since it is elementary (it involves only an estimation of the upper bound for $|\int_0^\infty (\mathrm{e}^{-\beta t} - \mathrm{e}_N^{-\beta t}) g(t)\,\mathrm{d}t|$, where the function $\mathrm{e}_N^{-\beta t}$ is a sum of the first N terms of the Taylor series of $\mathrm{e}^{-\beta t}$).

In the second column of table 1 we represent the numerical values of the function $f(t)$ computed for different t-values with the help of series (8). The computations have been performed with the help of 'Mathematica'. For the coefficients c_n we used the data from Janke and Kleinert (1995) where the first 22 coefficients of the strong-coupling expansion for the ground-state energy of the quartic anharmonic oscillator are given. Since the normalization of the coupling constant used in this paper is different from that used by Janke and Kleinert (1995) (these authors write the anharmonicity term in the Hamiltonian (1) as $gx^4/4$ instead of ours $\beta x^4/2$, and calculate the coefficients of the strong-coupling expansion in powers of $g^{-\frac{2}{3}}$), the coefficients c_n from equation (3) are actually the coefficients α_n of the strong-coupling expansion given by Janke and Kleinert (1995) multiplied by a factor of $2^{\frac{2n-1}{3}}$. For the sake of completeness we present in the second column of table 2 the first 22 coefficients of series (3). To save space only a few digits of the coefficients c_n are presented in the table.

Due to the rapid convergence of the series, we were able to calculate the function $f(t)$ up to rather large values of t. An inspection of the summation process when calculating the sum in equation (8) shows that account of the first 22 terms of series (8) allows us to reach an accuracy of about 1 part in 10^4 even for t as large as 40. For smaller values of t the accuracy is much higher. Thus, we estimate, that for $t \approx 25$ account of the first 22 terms of series (8) yields an accuracy of one part in 10^6, and for $t \approx 15$ to one part in 10^9.

The data from the second column of table 1 show that our hypothesis (10) is very likely to be true. Having established this fact we may proceed further and obtain some additional

Table 2. Coefficients c_n and sequence $|c_n|^{1/n}$

n	c_n	$\|c_n\|^{1/n}$
0	0.530 181 0452	
1	0.181 011 3244	0.181 011
2	−0.017 255 1314	0.131 359
3	0.002 597 6514	0.137 465
4	−0.000 415 4172	0.142 765
5	0.000 064 5560	0.145 207
6	$-9.244\,732 \times 10^{-6}$	0.144 871
7	$1.131\,832 \times 10^{-6}$	0.141 430
8	$-9.438\,601 \times 10^{-8}$	0.132 393
9	$-3.261\,936 \times 10^{-9}$	0.114 039
10	$3.887\,755 \times 10^{-9}$	0.144 202
11	$-1.144\,361 \times 10^{-9}$	0.153 866
12	$2.449\,702 \times 10^{-10}$	0.158 159
13	$-4.204\,230 \times 10^{-11}$	0.159 156
14	$5.509\,166 \times 10^{-12}$	0.156 961
15	$-3.623\,969 \times 10^{-13}$	0.148 120
16	$-7.610\,623 \times 10^{-14}$	0.151 387
17	$3.931\,341 \times 10^{-14}$	0.162 721
18	$-1.068\,970 \times 10^{-14}$	0.167 429
19	$2.217\,113 \times 10^{-15}$	0.169 327
20	$-3.640\,361 \times 10^{-16}$	0.169 066
21	$4.200\,939 \times 10^{-17}$	0.166 020
22	$-6.481\,760 \times 10^{-19}$	0.149 025

information about the function $g(t)$. To do that we shall study the asymptotic behaviour of the perturbation coefficients E_n following from equation (11) and compare it with the asymptotic formula (4) by Bender and Wu. One can see from equation (11) that when $n \to \infty$ it is the region of the large t-values which contributes to the integral. We therefore need a guess about the large-t asymptotic behaviour of the function $g(t)$. Let us suppose that for large t-values the function $g(t)$ behaves as

$$g(t) \sim At^{\alpha} \exp(-bt^{\gamma}) \tag{12}$$

where A, α, b, γ are some constants to be determined. Substituting this expression into equation (11) and performing the integration one obtains for $n \to +\infty$

$$E_n \sim \frac{(-1)^{n-1}}{(n-1)!} \frac{A}{\gamma} \frac{\Gamma(\frac{\alpha+n}{\gamma})}{b^{\frac{\alpha+n}{\gamma}}}. \tag{13}$$

Using the known properties of gamma functions (Abramovitz and Stegun 1964) one can easily show that this expression coincides with the leading term of the true asymptotic behaviour of the coefficients E_n for the ground-state energy of the quartic anharmonic oscillator, given by equation (4) if, and only if, the parameters in equation (12) are chosen as $\alpha = 0$, $\gamma = \frac{1}{2}$, $b = \sqrt{\frac{8}{3}}$, $A = \sqrt{6}/\pi$. To reproduce the next-to-leading-order terms of the asymptotic formula (4), one should suppose that when $t \to +\infty$ the corrections to the large-t asymptotic behaviour (12) of the function $g(t)$ are of the form

$$g(t) \sim \frac{\sqrt{6}}{\pi} \exp(-\sqrt{8t/3}) \left(1 - \frac{e_1}{\sqrt{t}} - \frac{e_2}{t} - \frac{e_3}{t^{\frac{3}{2}}} \ldots\right) \tag{14}$$

with the coefficients e_i to be determined. Substituting expression (14) into equation (11), integrating and comparing the result with the asymptotic formula (4) one can find the values of the coefficients e_i on the right-hand side of equation (14). For the first two coefficients e_i one obtains $e_1 = a_1\sqrt{\frac{3}{2}}$, $e_2 = (6a_2 - 3a_1)/4$, where a_1, a_2 are the coefficients on the right-hand side of the formula (4) by Bender and Wu. Thus, to reproduce the correct large-n asymptotic behaviour of the coefficients E_n for the ground-state energy, the function $f(t)$ given by equation (8) must have the following large-t asymptotic behaviour

$$\sum_{m=0}^{\infty} \frac{c_m}{\Gamma(\frac{2m+2}{3})} t^{\frac{2m-1}{3}} \sim E_0 + \frac{\sqrt{6}}{\pi} \exp\left(-\sqrt{8t/3}\right) \left(1 - \frac{a_1\sqrt{\frac{3}{2}}}{\sqrt{t}} - \frac{6a_2 - 3a_1}{4t} \ldots\right) \qquad (15)$$

where $E_0 = \frac{1}{2}$, $a_1 = \frac{95}{72}$ and $a_2 = 20\,099/10\,368$. We emphasize that this expression has no fitting parameters. Once the functional form (12), (14) of the asymptotic behaviour of the function $g(t)$ is adopted, all the parameters on the right-hand side of formula (15) are determined uniquely. Formula (15) can easily be verified. In the third column of table 1 we present numerical values given by the asymptotic formula (15) provided that only the two leading terms of expansion (15) are taken into account (i.e. the terms of orders $t^{-\frac{1}{2}}$ and t^{-1} in the parentheses on the right-hand side of equation (15) are omitted, this approximation is designated as $f^{(2)}(t)$ in table 1). The fourth column of table 1 contains the results obtained if three leading terms of expansion (15) are taken into account (i.e. the term of order $t^{-\frac{1}{2}}$ is kept in the parentheses on the right-hand side of equation (15), this approximation is designated as $f^{(3)}(t)$ in the table), and the fifth column of table 1 contains the results obtained with account of the four leading terms of equation (15) (the terms of orders $t^{-\frac{1}{2}}$, t^{-1} are kept in the parentheses on the right-hand side of equation (15), designated as $f^{(4)}(t)$ in table 1). We recall that the second column of this table contains the numerical values of the function $f(t)$ calculated according to equation (8).

The comparison of the data strongly supports the conclusion that formula (15) describes correctly the large-t asymptotic behaviour of the function $f(t)$ given by equation (8) and that representation (5) with the function $f(t)$ given by equation (8) is valid. Both strong-coupling and weak-coupling expansions can be obtained from equation (5) as two limiting cases.

Let us discuss formula (15) in more detail. The left-hand side of it is series (8) with the coefficients directly related to the coefficients of the strong-coupling expansion (3). The right-hand side is known in analytical form. There may exist a way of obtaining an analytical information about the asymptotic large-n behaviour of the coefficients c_n from this expression. We did not find a direct way to do it. Instead, we performed a numerical study of series (8) which allowed us to obtain some information about c_n indirectly. This study is reported below.

2.2. Numerical study of the coefficients c_n

We shall need a few facts from the theory of entire functions. First, we recall the definitions of such concepts as order and type of an entire function of finite order and type (Markushevitch 1968). For a given entire function $h(u)$ a function $M(r)$ is introduced as

$$M(r) = \max_{|u|=r} |h(u)|. \qquad (16)$$

According to well known properties of analytic functions $M(r)$ is a monotonous growing function of r. Omitting some unnecessary details, order ρ and type σ of an entire function $h(u)$ can be defined as the smallest possible non-negative numbers for which $M(r) <$

$\exp(\sigma r^{\rho})$ for all sufficiently large r. Roughly speaking, these numbers determine how rapidly the maximum of the absolute value of $h(u)$ grows. One can show (Markushevitch 1968) that order ρ of an entire function represented by a series $\sum b_n u^n$, can be expressed through the coefficients b_n of its power series expansion as

$$\rho = -\limsup_{n\to\infty} \frac{n \ln n}{\ln |b_n|}. \tag{17}$$

Let us consider the series obtained from series (8) by making a substitution $t^{2/3} = u$

$$h(u) = \sum_{m=0}^{\infty} \frac{c_m}{\Gamma(\frac{2m+2}{3})} u^m. \tag{18}$$

As we saw when discussing the convergence properties of series (8), $h(u)$ is an entire function of u. It can be seen that the order of this function is $\rho = \frac{3}{2}$. Indeed, substituting the coefficients of expansion (18) into equation (17), using the asymptotic expression for the gamma function and taking into account that according to the Hadamard formula $|c_n|^{1/n}$ has a finite limit, one can see that for the function defined by series (18), $\rho = \frac{3}{2}$. If we knew the type σ of this function the radius of convergence of series (18) could be found as follows.

We shall need one more result from the theory of entire functions (Markushevitch 1968), stating that if b_n are the coefficients of a power series expansion of an entire function having finite-order ρ and type σ, then the following equality holds

$$\limsup_{n\to\infty} n^{1/\rho} |b_n|^{1/n} = (e\rho\sigma)^{1/\rho}. \tag{19}$$

Applying this formula for the entire function (18), taking into account the Hadamard formula (9) and using asymptotic expression for the gamma function (Abramovitz and Stegun 1964), one obtains

$$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = \frac{1}{R} = \sigma^{\frac{2}{3}} \tag{20}$$

where R is the radius of convergence of the series $\sum c_n u^n$ and σ is the type of the entire function (18).

To find the value of σ for the function $h(u)$ given by series (18) we adopted the following strategy. We computed the sum of series (18) at the points lying on the circles $|u| = r_i$ in the u-complex plane with increasing r_i. We found that for $6 < |u| < 12$ the maximum of the absolute value of $|h(u)|$, for u lying on a circle $|u| =$ constant, is always achieved when u is on the negative real axis. In the second column of table 3 we present the values of the $M(r_i) = \max_{|u|=r_i} |h(u)|$ for $r_i = 6$–12. In the third column of this table we present the sequence of the ratios $\sigma_i = \ln M(r)/r^{3/2}$ which according to the definition of a type of an entire function, converges to σ.

We arrive at the following picture. For $u \to +\infty$ along the positive real axis the behaviour of $h(u)$ is known analytically and can be obtained from equation (15). On the negative real axis $h(u)$ grows exponentially as $\exp(\sigma u^{3/2})$ with $\sigma \approx 0.068$.

In the fourth column of table 3 we present the numbers from the preceding column raised to the power $\frac{2}{3}$. According to equation (20) the limit of this sequence is equal to $1/R$, where R is the radius of convergence of the series $\sum c_n u^n$. According to the Hadamard formula the same quantity can be found as a limit of the sequence $|c_n|^{1/n}$. In the third column of table 2 we present this sequence. These numbers are rather close to those from the fourth column of table 3 but have somewhat greater dispersion. This is the reason why we used the procedure based on the study of the function $h(u)$ for the estimation of the radius of

Table 3. $M(r_i) = \max_{u=r_i} |h(u)|$ as a function of r_i and σ_i.

r_i	$M(r_i)$	σ_i	$\sigma_i^{2/3}$
6	2.134 42	0.051 5886	0.138 5806
7	3.104 81	0.061 1737	0.155 2541
8	4.413 17	0.065 6104	0.162 6729
8	6.182 72	0.067 4725	0.165 7364
10	8.569 24	0.067 9314	0.166 4870
11	11.7572	0.067 5513	0.165 8654
12	15.9419	0.066 6106	0.164 3220

Table 4. The function $f(t)$ for the first excited state of the quartic oscillator.

t	$f(t)$
1	1.697 459
2	1.586 290
3	1.547 105
4	1.527 804
5	1.516 225
6	1.508 012
7	1.501 209

convergence. We found that it gives better numerical results, which we would attribute to the fact, that the numerical study of the function given by series (18) is less sensitive to small numerical uncertainties in the numerical values of the large-order coefficients c_n.

From the numbers in the third column of table 3 one can estimate the radius of convergence of the series (18) as $R \approx \frac{1}{0.17} = 5.9$. This is the radius of convergence of the series in powers of $u = 1/\beta^{2/3}$. In the β-complex plane we obtain thus an estimation that the strong-coupling expansion (3) for the ground-state energy of the quartic anharmonic oscillator converges if $|\beta| > 0.07$.

3. Remarks

For the ground-state energy $E(\beta)$ of the quartic anharmonic oscillator we have given a representation reproducing both limiting cases of the theory: the strong-coupling and the weak-coupling regimes. It was shown, that the asymptotic behaviour of function (8) given by a power series with the coefficients depending upon the strong-coupling coefficients (3) can be obtained in a closed analytical form. An interesting question is: can one extract analytical information about the coefficients c_n from this relation?

The representation analogous to (5) can also be constructed for the excited states of the quartic anharmonic oscillator. In table 4 we present some results for the first excited state of the quartic anharmonic oscillator. We used the data for the first five coefficients of the strong-coupling expansion for the first excited state, given in the work by Fernandez (1992). In the second column of table 4 we represent the sum of the first five terms of series (8). One can see, that for the large t-values the sum of the series tends to $\frac{3}{2}$ as could be expected on the basis of formula (10).

One can give another derivation of the asymptotic formula (15) which reveals the well known connection between the behaviour of perturbation theory at large orders and barrierpenetration rate (Le Guillou and Zinn-Justin 1990). Let us deform the contour of integration in equation (6) so that it starts at $-\infty$, goes along the negative real in the lower half-plane, turns anticlockwise round the origin and goes to $-\infty$ along the negative real axis in the upper half-plane. If we are interested in the large-t asymptotic behaviour of the function $f(t)$ it is easy to see that it is a region near the origin which contributes mostly to the integral (6). Consider separately the contributions to the integral (6) due to the real and imaginary parts of energy. One can see that the contribution of the real part can be calculated just by replacing $\mathrm{Re}\,E(\beta)$ by its value for $\beta = 0$, i.e. by $\frac{1}{2}$. A simple integration yields $\frac{1}{2}$. After simple manipulations, formula (6) can be rewritten as

$$f(t) = \frac{1}{2} - \frac{1}{\pi} \int_{-\infty}^{0} \frac{\mathrm{Im}\,E(\beta)}{\beta} \mathrm{e}^{\beta t}\, \mathrm{d}\beta \tag{21}$$

where we used the fact that $E(\beta)$ assumes complex conjugate values on the upper and lower sides of the cut. To estimate the latter integral in the limit of large positive t one should know the behaviour of $\mathrm{Im}\,E(\beta)$ for small negative β-values. It is given by the well known semiclassical formula

$$\mathrm{Im}\,E^{sc}(\beta) = \sqrt{-\frac{4}{\pi\beta}} \exp\left\{\frac{2}{3\beta}\right\}. \tag{22}$$

Substituting the latter equation into formula (21) and calculating the integral with the help of the saddle-point method, one recovers the asymptotic formula (15) for the function $f(t)$.

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